

A REMARK ON $C^{2,\alpha}$ -REGULARITY OF THE COMPLEX MONGE-AMPÈRE EQUATION

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ABSTRACT. We prove the $C^{2,\alpha}$ -regularity of the solution u of the equation

$$\det(u_{\bar{k}j}) = f, \quad f^{1/n} \in C^\alpha, \quad f \geq \lambda$$

under the assumption in upper bound of Δu . Our result settles down the regularity problem related to the paper [3] (also see [4]).

1. INTRODUCTION

In this note, we prove

Theorem 1.1. *Let $u \in C^2(B_1)$ be a plurisubharmonic function that solves the equation*

$$(1.1) \quad \det(u_{\bar{k}j}) = f \quad \text{in } B_1.$$

Suppose $0 < \alpha < 1$

$$(1.2) \quad f^{1/n} \in C^\alpha(\overline{B_1}), \quad \inf_{B_1} f^{1/n} \geq \lambda > 0$$

and

$$(1.3) \quad \sup_{B_1} \Delta u \leq \Lambda.$$

Then, for any $0 < \beta < \alpha$, there exists a constant C depending on $\beta, n, \lambda, \Lambda, \|f^{1/n}\|_{C^\alpha(\overline{B_1})}, \|u\|_{L^\infty(B_1)}$ such that

$$u \in C^{2,\beta}(\overline{B_{1/2}}) \quad \text{and} \quad \|u\|_{C^{2,\beta}(\overline{B_{1/2}})} \leq C.$$

By the standard nonlinear elliptic theory (Thm.3 in [2] and Thm.6.6 in [1]), the above regularity result is a direct consequence of the following theorem:

Theorem 1.2. *Let $u \in C^2(B_1)$ be a plurisubharmonic function that solves the equation (1.1). Suppose*

$$(1.4) \quad \inf_{B_1} f^{1/n} \geq \lambda > 0, \quad \text{and} \quad \sup_{B_1} \Delta u \leq \Lambda.$$

Then, there exists a concave function \tilde{F} on the space of $2n \times 2n$ real symmetric matrices such that

i) \tilde{F} is θ -uniform elliptic, i.e.,

$$\theta \|P\| \leq \tilde{F}(M + P) - \tilde{F}(M) \leq \theta^{-1} \|P\|, \quad \forall M \in \text{Sym}(2n), P \geq 0$$

where θ only depends on λ, Λ, n .

ii) u satisfies the equation

$$\tilde{F}(D^2u) = f^{1/n} \quad \text{in } B_1.$$

Priori to this note, the best result in this direction is obtained by S. Dinew, X. Zhang and X.-W Zhang [4]. They have proved $C^{2,\alpha}$ -regularity under the assumption of the L^∞ -bound of D^2u . Their proof is based on the perturbation argument developed by Trudinger and Wang [5], [6].

Contrary to the method employed in [4], we reduce the problem to the uniform elliptic case by constructing an suitable extension of determinant outside a certain set in the space of matrices. The idea of this note is suggested by Prof. Ovidiu Savin.

2. THE PROOF OF THM.1.2

Let $\text{Sym}(2n)$ be the space of $2n \times 2n$ real symmetric matrices and $\text{Herm}(n)$ be the space of $n \times n$ complex Hermitian matrices.

Fix the following canonical complex structure

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad I \text{ is the } n \times n \text{ identity matrix}$$

on \mathbb{R}^{2n} . Then $\text{Herm}(n)$ can be identified to the subspace

$$\{M : [M, J] = MJ - JM = 0\} \subset \text{Sym}(2n)$$

by the map

$$\iota : H = A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

In the rest of this note, we always view $\text{Herm}(n)$ as a subspace of $\text{Sym}(2n)$ according to the above identification.

The complex structure J gives rise the canonical projection $p : \text{Sym}(2n) \rightarrow \text{Herm}(n)$

$$p : M \mapsto \frac{M + J^t M J}{2}.$$

The complex determinant $\det_{\mathbb{C}}$ on Hermitian matrices is related to the real determinant $\det_{\mathbb{R}}$ by

$$\det_{\mathbb{R}}^{1/2n}[p(M)] = \det_{\mathbb{C}}^{1/n}[H] \quad \text{if } \iota(H) = p(M), \quad M \in \text{Sym}(2n), \quad H \in \text{Herm}(n).$$

We denote

$$(2.1) \quad F(M) := \det_{\mathbb{R}}^{1/2n}[p(M)], \quad M \in \text{Sym}(2n).$$

By the Minkowski inequality, F is a concave function on the set

$$\{M \in \text{Sym}(2n) : p(M) > 0\}.$$

Now, we give the construction of \tilde{F} :

Definition 2.1. Given $\theta > 0$, let $\mathcal{E}_{\theta} \subset \text{Sym}(2n)$ consist of matrices N such that

$$\theta I \leq p(N) \leq \theta^{-1} I.$$

Define, for all $M \in \text{Sym}(2n)$,

$$\tilde{F}(M) := \inf\{\text{tr}[p(N)M] + c : \text{tr}[p(N)X] + c \geq F(X) \quad \forall X \in \mathcal{E}_{\theta} \quad \text{ii) } N \in \mathcal{E}_{\theta}, c \in \mathbb{R}\}.$$

Remark 2.2. \tilde{F} is the concave envelop of F over the set \mathcal{E}_{θ} .

Remark 2.3. The above construction is suggested by Prof. Ovidiu Savin. The author's original approach is to extend level set of F outside \mathcal{E}_{θ} . Though it will give essentially same function as above, the construction in Def.2.1 is more direct and transparent.

The following lemma is the main ingredient in proving Thm.1.2;

Lemma 2.4. \tilde{F} is concave and uniformly elliptic in $\text{Sym}(2n)$, i.e., there exists $\tilde{\theta} > 0$ only depends on θ such that

$$(2.2) \quad \tilde{\theta}\|P\| \leq F(M+P) - F(M) \leq \tilde{\theta}^{-1}\|P\|, \quad \forall M \in \text{Sym}(2n), P \geq 0$$

Moreover, $\tilde{F}(M) = F(M)$ for all $M \in \mathcal{E}_{\theta}$.

Proof. Concavity and agreement on \mathcal{E}_θ follow directly from the construction. We only need to check the ellipticity. Given $M \in \text{Sym}(2n)$, $P \geq 0$, by the definition of \tilde{F} , there are $N_1, N_2 \in \mathcal{E}_\theta$ such that

$$\tilde{F}(M + P) = \text{tr}[p(N_1)(M + P)] + c_1$$

and

$$\tilde{F}(M) = \text{tr}[p(N_2)(M)] + c_2.$$

By the minimality, we have

$$\text{tr}[p(N_1)M] + c_1 \geq \text{tr}[p(N_2)M] + c_2$$

and

$$\text{tr}[p(N_1)(M + P)] + c_1 \leq \text{tr}[p(N_2)(M + P)] + c_2.$$

Then, combine above inequalities, we have

$$\tilde{F}(M + P) - \tilde{F}(M) \geq \text{tr}[p(N_1)(M + P)] - \text{tr}[p(N_1)(M)] \geq \frac{\theta}{4n} \|P\|$$

and

$$\tilde{F}(M + P) - \tilde{F}(M) \leq \text{tr}[p(N_2)(M + P)] - \text{tr}[p(N_2)M] \leq 2n\theta^{-1} \|P\|.$$

This completes the proof of the lemma. \square

Now, we are ready to complete the proof of Thm.1.2. Since $u \in C^2(B_1)$ satisfies (1.4), we have

$$\lambda I \leq p(D^2u)(x) \leq \Lambda I, \quad \forall x \in B_1.$$

In turn, by taking $\theta = \min\{\lambda, \Lambda^{-1}\}$, we have

$$(2.3) \quad D^2u(x) \in \mathcal{E}_\theta, \quad \forall x \in B_1.$$

Now consider \tilde{F} given by the Def.2.1 with respect to \mathcal{E}_θ . By (2.3), Lem.2.4 and (2.1)

$$\tilde{F}(D^2u(x)) = F(D^2u(x)) = f^{1/n}(x) \quad x \in B_{1/2}.$$

Uniform ellipticity and concavity of \tilde{F} have been given in Lem.2.4. This completes the proof of Thm.1.2.

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